NOTE ON THE FINITE CONTINUOUS GROUPS OF THE PLANE.

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Presented by Henry Taber, October 11, 1899.

SINCE Professor Study * made the important discovery that the special linear homogeneous group contains *singular transformations*, i. e. transformations that cannot be generated by an infinitesimal transformation of this group (in consequence of which the group is not continuous except in the neighborhood of the identical transformation), such singular transformations have been found by Professor Taber † and others, in many other sub-groups of the general projective group. Thus, e.g., Mr. Rettger has shown that of the 76 two and three term sub-groups of the projective group in two variables, and of the general linear homogeneous group in three variables, 21 contain singular transformations.‡ It was therefore to be expected that, for example, among the groups of the plane given by Lie on pages 360 and 361 of his *Continuierliche Gruppen*, some, not sub-groups of the projective group, would be found to contain singular transformations. This I find to be the case, as the second group considered below will show. The first group considered is projective for the value of r taken; and, in connection with the consideration of this group, there is given a method by means of which we are able to ascertain whether a group contains singular transformations or not.

Throughout this paper $p \equiv \frac{\partial}{\partial x}$ and $q \equiv \frac{\partial}{\partial y}$.

Example I.

If in the case of the group

$$q, \quad xq, \quad x^2q \ldots x^{r-3}q, \quad p, \quad xp + \alpha yq, \quad (r > 3)$$

* Leipziger Berichte, 1892.

† Bull. N. Y. Math. Soc., July, 1894; Math. Ann., Vol. XLVI. p. 561; Math. Review, Vol. I. p. 154. See also Newson, Kansas Univ. Quart., 1896.

‡ See These Proceedings, Vol. XXXIII.

we put $r = 4$, we have the group

$$q, \quad x q, \quad p, \quad xp + \alpha y q,$$

which is a sub-group of the general projective group.

The symbol of the general infinitesimal transformation is

$$U \equiv (a_1 + a_2 x + a_4 \alpha y) q + (a_3 + a_4 x) p.$$

Hence,

$$Ux = a_3 + a_4 x,$$

$$U^2 x = a_3 a_4 + a_4^2 x,$$

$$\cdots\cdots\cdots\cdot$$

$$U^n x = a_3 a_4^{n-1} + a_4^n x,$$

where $U^2 x$ denotes $U(Ux)$, etc. Similarly,

$$Uy = a_1 + a_2 x + a_4 \alpha y,$$

$$U^2 y = a_1 a_4 \alpha + a_2 a_4 \alpha x + a_4^2 \alpha^2 y + a_2 a_4 x + a_2 a_3,$$

$$U^3 y = a_1 a_4^2 \alpha^2 + a_2 a_4^2 \alpha^2 x + a_4^3 \alpha^3 y + a_2 a_4^2 (\alpha + 1) x + a_2 a_3 a_4 (\alpha + 1),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdot$$

$$U^n y = a_1 a_4^{n-1} \alpha^{n-1} + a_2 a_4^{n-1} \alpha^{n-1} x + a_4^n \alpha^n y + a_2 a_4^{n-1} (\alpha^{n-2} + \alpha^{n-3} + \ldots \alpha + 1) x$$
$$+ a_2 a_3 a_4^{n-2} (\alpha^{n-2} + \alpha^{n-3} + \ldots + \alpha + 1).$$

Therefore,* the transformation of the group generated by the general infinitesimal transformation of the group is defined by the equations

$$(1) \quad \begin{aligned} x' &= x e^{a_4} + \frac{a_3}{a_4} (e^{a_4} - 1), \\ y' &= y e^{\alpha a_4} + \frac{a_2 x}{a_4 (\alpha - 1)} (e^{\alpha a_4} - e^{a_4}) \\ &\quad + \frac{a_2 a_3}{\alpha a_4^2 (\alpha - 1)} (e^{\alpha a_4} - \alpha e^{a_4} - 1 + \alpha) + \frac{a_1}{\alpha a_4} (e^{\alpha a_4} - 1). \end{aligned}$$

Let this transformation be denoted by T_a. It transforms the point P with coördinates (x, y) into the point P' with coördinates (x', y'). Let the transformation T_b of our group (generated by the infinitesimal transformation $(b_1 + b_2 x + b_4 \alpha y) q + (b_3 + b_4 x) p$) transform P' into P'' with coördinates (x'', y''). T_b is then defined by the equations

* Lie, Differentialgleichungen, chap. 3, § 3.

$$(2)\qquad \begin{aligned} x'' &= x' e^{b_4} + \frac{b_3}{b_4}(e^{b_4} - 1), \\ y'' &= y' e^{\alpha b_4} + \frac{b_2 x}{b_4(\alpha - 1)}(e^{\alpha b_4} - e^{b_4}) \\ &\quad + \frac{b_2 b_3}{\alpha b_4^2(\alpha - 1)}(e^{\alpha b_4} - \alpha e^{b_4} - 1 + \alpha) + \frac{b_1}{\alpha b_4}(e^{\alpha b_4} - 1). \end{aligned}$$

Substituting in (2) the values of x' and y' from (1), we get the transformation $T_b T'_a$ which carries the point P into the point P''. The equations defining $T_b T_a$ are then

$$(3)\qquad \begin{aligned} x'' &= x e^{a_4 + b_4} + \frac{a_3}{a_4}(e^{a_4 + b_4} - e^{b_4}) + \frac{b_3}{b_4}(e^{b_4} - 1), \\ y'' &= y e^{\alpha(a_4 + b_4)} + \frac{x}{\alpha - 1}\left\{\frac{a_2}{a_4}(e^{\alpha(a_4 + b_4)} - e^{a_4 + \alpha b_4}) + \frac{b_2}{b_4}(e^{\alpha b_4} - e^{b_4})\right\} \\ &\quad + \frac{a_2 a_3}{\alpha a_4^2(\alpha - 1)}(e^{\alpha(a_4 + b_4)} - \alpha e^{a_4 + \alpha b_4} - e^{\alpha b_4} + \alpha e^{\alpha b_4}) + \frac{a_1}{\alpha a_4}(e^{\alpha(a_4 + b_4)} - e^{\alpha b_4}) \\ &\quad + \frac{b_2 b_3}{\alpha b_4^2(\alpha - 1)}(e^{\alpha b_4} - \alpha e^{b_4} - 1 + \alpha) + \frac{b_1}{\alpha b_4}(e^{\alpha b_4} - 1). \end{aligned}$$

If now $T_b T_a$, which is also a transformation of our group, is equivalent to a transformation T_c, generated by the infinitesimal transformation $(c_1 + c_2 x + c_4 \alpha y) q + (c_3 + c_4 x) p$, — that is, if $T_b T_a = T_c$, we have also

$$(4)\qquad \begin{aligned} x'' &= x e^{c_4} + \frac{c_3}{c_4}(e^{c_4} - 1), \\ y'' &= y e^{\alpha c_4} + \frac{c_2 x}{c_4(\alpha - 1)}(e^{\alpha c_4} - e^{c_4}) + \frac{c_2 c_3}{\alpha c_4^2(\alpha - 1)}(e^{\alpha c_4} - \alpha e^{c_4} - 1 + \alpha) \\ &\quad + \frac{c_1}{\alpha c_4}(e^{\alpha c_4} - 1). \end{aligned}$$

Whence, first,

$$e^{c_4} = e^{a_4 + b_4},$$

$$e^{\alpha c_4} = e^{\alpha(a_4 + b_4)};$$

and therefore,

$$c_4 = a_4 + b_4 + 2\kappa\pi i,$$

$$\alpha c_4 = \alpha(a_4 + b_4) + 2\kappa'\pi i,$$

where κ and κ' are integers. From these equations it follows that $\alpha\kappa$ is an integer. Therefore, if α is irrational, $\kappa = 0$. On the other hand, if α is rational and equal to $\frac{\mu}{\nu}$, where μ and ν are integers relatively prime, $\kappa = \lambda\nu$, where λ is an arbitrary integer.

We also derive from (3) and (4)

$$c_3 = \frac{a_4 + b_4 + 2\kappa\pi i}{e^{a_4+b_4} - 1}\left\{\frac{a_3}{a_4}(e^{a_4+b_4} - e^{b_4}) + \frac{b_3}{b_4}(e^{b_4} - 1)\right\}$$
$$\equiv \psi(a, b),$$

$$c_2 = \frac{a_4 + b_4 + 2\kappa\pi i}{e^{\alpha(a_4+b_4)} - e^{a_4+b_4}}\left\{\frac{a_2}{a_4}(e^{\alpha(a_4+b_4)} - e^{a_4+\alpha b_4}) + \frac{b_2}{b_4}(e^{\alpha b_4} - e^{b_4})\right\}$$
$$\equiv \phi(a, b),$$

$$c_1 = \frac{a_4 + b_4 + 2\kappa\pi i}{e^{\alpha(a_4+b_4)} - 1}\left\{-\frac{\phi\,\psi}{(a_4 + b_4 + 2\kappa\pi i)^2(\alpha - 1)}(e^{\alpha(a_4+b_4)}\right.$$
$$-\alpha e^{a_4+b_4} - 1 + \alpha) + \frac{a_2 a_3}{a_4^2(\alpha - 1)}(e^{\alpha(a_4+b_4)} - \alpha e^{a_4+\alpha b_4} - e^{\alpha b_4}$$
$$+ \alpha e^{\alpha b_4}) + \frac{a_1}{a_4}(e^{\alpha(a_4+b_4)} - e^{\alpha b_4}) + \frac{b_2 b_3}{b_4^2(\alpha - 1)}(e^{\alpha b_4} - \alpha e^{b_4}$$
$$\left. - 1 + \alpha) + \frac{b_1}{b_4}(e^{\alpha b_4} - 1)\right\}.$$

If for finite values of the a's and b's, while some of the c's may remain finite, one (or more) becomes infinite in all branches, there is no infinitesimal transformation of the group that will generate $T_b\,T_a$, i. e. $T_b\,T_a$ is a singular transformation.

Let α be irrational. Then $\kappa = 0$; and for all finite values of the a's and b's, c_4 is finite. But, if $a_4 + b_4 = 2m\pi i$ for some integer $m \neq 0$, c_3 is infinite, provided

$$\frac{a_3}{a_4}(e^{a_4+b_4} - e^{b_4}) + \frac{b_3}{b_4}(e^{b_4} - 1) = \left(\frac{a_3}{a_4} - \frac{b_3}{b_4}\right)(1 - e^{b_4}) \neq 0.$$

Similarly, if $(\alpha - 1)(a_4 + b_4) = 2m\pi i \neq 0$, c_2 is in general infinite; and, if $\alpha(a_4 + b_4) = 2m\pi i \neq 0$, c_1 is in general infinite.

Let now $\alpha = \frac{\mu}{\nu}$; then c_4 is, as before, finite. In this case, as stated above, $\kappa = \lambda\nu$, where λ is an arbitrary integer; and if $a_4 + b_4 = 2m\pi i \neq 0$, c_3, and therefore c_1, are in general infinite unless

$$2\pi i(m + \lambda\nu) = a_4 + b_4 + 2\kappa\pi i = 0, —$$

that is, unless m contains ν. Therefore, if $a_4 + b_4 = 2\,m\,\pi\,i \neq 0$ and $\nu \neq 1$, we can always so choose m that c_3 shall, in general, be infinite and $T_b\,T_a$ singular. On the other hand, if $a_4 + b_4 = 2\,m\,\pi\,i$, and if $\nu = 1$ (i. e. if α is an integer), one branch of c_3 is always finite, and the same is true for c_1 and c_2: so in this case $T_b\,T_a$ can be generated by an infinitesimal transformation of our group.

When α is rational there are, however, always singular transformations of the group. For let

$$a_4 + b_4 = \frac{2\,m\,\pi\,i}{\alpha - 1} = \frac{2\,m\,\nu\,\pi\,i}{\mu - \nu}.$$

Then in general (i. e. provided the function of the a's and b's found in the second factor in the expression for c_2 is not zero), c_2 is infinite unless

$$2\,\nu\,\pi\,i\left(\frac{m}{\mu - \nu} + \lambda\right) = a_4 + b_4 + 2\,\kappa\,\pi\,i = 0\,;$$

which is impossible if m is so chosen that it shall not contain $\mu - \nu$.

Therefore, whether α is rational or irrational, if $a_4 + b_4 = \dfrac{2\,m\,\pi\,i}{\alpha - 1} \neq 0$ (where m is an integer which if α is rational and equal to $\dfrac{\mu}{\nu}$ does not contain $\mu - \nu$), c_2 is in general * infinite, and consequently $T_b\,T_a$ cannot be generated by an infinitesimal transformation of our group; i. e. $T_b\,T_a$ is then singular.

Among the singular transformations of our group obtained by putting $a_4 + b_4 = \dfrac{2\,m\,\pi\,i}{\alpha - 1} \neq 0$ (where if α is rational and equal to $\dfrac{\mu}{\nu}$, the integer m does not contain $\mu - \nu$), let us consider those for which, further, $a_3 = b_3 = 0$. These singular transformations are defined by the equations

$$(5)\qquad \begin{aligned} x' &= x\,e^{\frac{2\,m\,\pi\,i}{\alpha - 1}}, \\ y' &= y\,e^{\frac{2\,\alpha\,m\,\pi\,i}{\alpha - 1}} + M x + N \qquad (M \neq 0). \end{aligned}$$

The singular transformations T defined by equations (5) leave invariant, as a whole, the system of lines $x = $ const., but change each line into

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* I. e. provided $\dfrac{a_2}{a_4}\left(e^{\alpha(a_4 + b_4)} - e^{a_4 + \alpha b_4}\right) + \dfrac{b_2}{b_4}\left(e^{\alpha b_4} - e^{b_4}\right)$, which in this case becomes $\left(\dfrac{a_2}{a_4}\,e^{a_4} - \dfrac{b_2}{b_4}\right)\left(e^{b_4} - e^{\alpha b_4}\right)$, is not zero.

some other line of the system. Associated with T is a one-term group whose path curves, $x = c$, are as a whole unchanged by T. The path curves of our group are given by the equation

$$\frac{dx}{a_3 + a_4 x} = \frac{dy}{a_1 + a_2 x + a_4 \alpha y};$$

the solution of which gives

$$y = \frac{a_2}{a_4^2 (1 - \alpha)} (a_3 + a_4 x) - \frac{a_1 a_4 - a_2 a_3}{\alpha a_4^2} + \gamma (a_3 + a_4 x)^\alpha,$$

where γ is the constant of integration. If in the symbol of the general infinitesimal transformation U we put $a_4 = 0$, $a_3 = 0$, and a_1 and a_2 finite, we get the one-term group whose symbol of infinitesimal transformation is $U_1 \equiv (a_1 + a_2 x)\, q$, and whose path curves are $x =$ const.; which is then the one-term group associated with the singular transformations T.

Example II.

$$\boxed{\begin{array}{c} e^{\alpha_\kappa x} q, \quad x e^{\alpha_\kappa x} q, \ldots x^{\rho_\kappa} e^{\alpha_\kappa x} q, \quad p \\ \kappa = 1, 2, \ldots m, \quad \alpha_\kappa = \text{const.}, \quad \Sigma \rho_\kappa + m r = 1, \quad r > 2. \end{array}}$$

Put $r = 3$, $\kappa = m = 1$, and $\rho_\kappa = 1$. We then have the group

$$e^{\alpha x} q, \qquad x e^{\alpha x} q, \qquad p.$$
$$(\alpha \neq 0)$$

The symbol of the general infinitesimal transformation is

$$U \equiv (a_1 e^{\alpha x} + a_2 x e^{\alpha x})\, q + a_3 p.$$

Hence,

$$\begin{aligned} U x &= a_3, \\ U^2 x &= 0, \\ &\cdots\cdots \\ U^n x &= 0, \end{aligned}$$

and therefore $x' = x + a_3$. Further,

$$\begin{aligned} U y &= a_1 e^{\alpha x} + a_2 x e^{\alpha x}, \\ U^2 y &= a_1 a_3 \alpha e^{\alpha x} + a_2 a_3 e^{\alpha x} + a_2 a_3 \alpha x e^{\alpha x}, \\ U^3 y &= a_1 a_3^2 \alpha^2 e^{\alpha x} + 2 a_2 a_3^2 \alpha e^{\alpha x} + a_2 a_3^2 \alpha^2 x e^{\alpha x}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ U^{n+1} y &= a_1 a_3^n \alpha^n e^{\alpha x} + n a_2 a_3^n \alpha^{n-1} e^{\alpha x} + a_2 a_3^n \alpha^n x e^{\alpha x}. \end{aligned}$$

Therefore,

$$y' = y + x e^{\alpha x} \left\{ \frac{a_2}{\alpha a_3} (e^{\alpha a_3} - 1) \right\} + e^{\alpha x} \left[\frac{a_1}{\alpha a_3} (e^{\alpha a_3} - 1) \right.$$

$$\left. + a_2 \left\{ \frac{a_3}{2} + a_3 \left[\frac{\partial}{\partial a_3} \left\{ \frac{e^{\alpha a_3} - \left(1 + \alpha a_3 + \frac{\alpha^2 a_3^2}{2}\right)}{\alpha^2 a_3} \right\} \right] \right\} \right]$$

Hence the ∞^3 of non-singular transformations T_a have the form

$$(1) \qquad \begin{aligned} x' &= x + a_3, \\ y' &= y + x e^{\alpha x} \phi(a) + e^{\alpha x} \psi(a). \end{aligned}$$

where

$$\phi(a) \equiv a_2 \frac{e^{\alpha a_3} - 1}{\alpha a_3},$$

$$\psi(a) \equiv a_1 \frac{e^{\alpha a_3} - 1}{\alpha a_3} + a_2 \left\{ \frac{a_3}{2} + a_3 \left[\frac{\partial}{\partial a_3} \left\{ \frac{e^{\alpha a_3} - \left(1 + \alpha a_3 + \frac{\alpha_2 a_3^2}{2}\right)}{\alpha_2 a_3} \right\} \right] \right\}$$

$$\equiv a_1 \frac{e^{\alpha a_3} - 1}{\alpha a_3} + \frac{a_2}{\alpha} \left\{ e^{\alpha a_3} \left(1 - \frac{1}{\alpha a_3}\right) + \frac{1}{\alpha a_3} - \frac{\alpha a_3}{2} \right\}.$$

Similarly, the transformation T_b may be defined by the equations

$$(2) \qquad \begin{aligned} x'' &= x' + b_3, \\ y'' &= y' + x e^{\alpha x'} \phi(b) + e^{\alpha x'} \psi(b). \end{aligned}$$

And, therefore, $T_b\, T_a$ is defined by the equations

$$(3) \qquad \begin{aligned} x'' &= x + a_3 + b_3, \\ y'' &= y + x e^{\alpha x} [\phi(a) + e^{\alpha a_3} \phi(b)) + e^{\alpha x} (\psi(a) + a_3 e^{\alpha a_3} \phi(b) \\ &\quad + e^{\alpha a_3} \psi(b)]. \end{aligned}$$

If now $T_b\, T_a = T_c$, we have also

$$(4) \qquad \begin{aligned} x'' &= x + c_3, \\ y'' &= y + x e^{\alpha x} \phi(c) + e^{\alpha x} \psi(c). \end{aligned}$$

Therefore,

$$\begin{aligned} c_3 &= a_3 + b_3, \\ \phi(c) &= \phi(a) + e^{\alpha a_3} \phi(b), \\ \psi(c) &= \psi(a) + a_3 e^{\alpha a_3} \phi(b) + e^{\alpha a_3} \psi(b). \end{aligned}$$

Whence we derive

$$c_3 = a_3 + b_3,$$

$$c_2 = \frac{\alpha\,(a_3 + b_3)}{e^{\alpha\,(a_3 + b_3)} - 1}\left[\frac{a_2}{\alpha\,a_3}\,(e^{\alpha\,a_3} - 1) + \frac{b_2}{\alpha\,b_3}\,(e^{\alpha\,(a_3 + b_3)} - e^{\alpha\,a_3})\right]$$
$$\equiv \Omega\,(a, b),$$

$$c_1 = \frac{\alpha\,(a_3 + b_3)}{e^{\alpha\,(a_3 + b_3)} - 1}\left[-\frac{\Omega\,(a, b)}{\alpha}\left\{e^{\alpha\,(a_3 + b_3)}\left(1 - \frac{1}{\alpha\,(a_3 + b_3)}\right)\right.\right.$$
$$\left.\left. + \frac{1}{\alpha\,(a_3 + b_3)} - \frac{\alpha\,(a_3 + b_3)}{2}\right\} + \psi\,(a) + a_3 e^{\alpha\,a_3}\phi\,(b) + e^{\alpha\,a_3}\psi(b)\right].$$

From these equations it follows that c_3 is finite for finite values of the a's and b's; but if $a_3 + b_3 = \frac{2\,\kappa\,\pi\,i}{\alpha} \neq 0$, where κ is an integer not zero (α being either rational or irrational), then c_1 and c_2 are, in general, both infinite in all branches (and, indeed, c_1 is infinite to the second order), that is, unless

$$(5) \qquad \frac{a_2}{\alpha\,a_3}\,(e^{\alpha\,a_3} - 1) + \frac{b_2}{\alpha\,b_3}\,(e^{\alpha\,(a_3 + b_3)} - e^{\alpha\,a_3}) = 0,$$

and

$$(6) \qquad \frac{-\,\Omega\,(a, b)}{\alpha}\left[e^{\alpha\,(a_3 + b_3)}\left(1 - \frac{1}{\alpha\,(a_3 + b_3)}\right) + \frac{1}{\alpha\,(a_3 + b_3)} - \frac{\alpha\,(a_3 + b_3)}{2}\right]$$
$$+ \psi\,(a) + a_3\,e^{\alpha\,a_3}\,\phi\,(b) + e^{\alpha\,a_3}\,\psi\,(b) = 0.$$

If (5) is satisfied and (6) is not, we have c_2 finite and c_1 infinite to the first order for $a_5 + b_3 = \frac{2\,\kappa\,\pi\,i}{\alpha} \neq 0$. Therefore, if $a_3 + b_3 = \frac{2\,\kappa\,\pi\,i}{\alpha} \neq 0$, where κ is an integer, $T_b\,T_a$ is, in general, singular.

Among the singular transformations of our group obtained by putting $a_3 + b_3 = \frac{2\,\kappa\,\pi\,i}{\alpha} \neq 0$ (κ an integer), let us consider those for which, further, $a_2 = b_2 = 0$. Equation (5) is then satisfied; and these singular transformations are defined by the equations

$$(7) \qquad \begin{aligned} x' &= x + \frac{2\,\kappa\,\pi\,i}{\alpha} && (\kappa \text{ an integer} \neq 0), \\ y' &= y + M e^{\alpha\,x} && (M \neq 0). \end{aligned}$$

The singular transformations T defined by equations (7) leave invariant, as a whole, the system of lines $x = \text{const.}$, but change each line into some other line of the system. Associated with T is a one-term group whose path curves, $x = c$, are as a whole unchanged by T. The path curves generated by the general infinitesimal transformation of this group are defined by the equation

$$\frac{dx}{a_3} = \frac{dy}{e^{\alpha x}(a_1 + a_2 x)};$$

the solution of which gives

$$e^{\alpha x}(a_1 \alpha + a_2 \alpha x - a_2) = a_3 \alpha^2 y + c. \qquad (c = \text{const.})$$

If, now, in the symbol of the general infinitesimal transformation U, we put $a_3 = 0$, $a_2 = 0$, and a_1 finite, we get the one-term group, whose symbol of infinitesimal transformation is $U_1 \equiv a_1 e^{\alpha x} q$, and whose path curves are $x = \text{const.}$; which is then the one-term group associated with the singular transformation T.

The following groups do not contain singular transformations, and are properly *continuous groups.*

Example III.

$$\boxed{\begin{matrix} q, \quad \phi_2(x)\, q, \; \ldots \; \phi_{r-1}(x)\, q, \quad y q \\ r > 2 \end{matrix}}$$

Put $r = 3$; we then have the group

$$q, \quad \phi(x)\, q, \quad y q;$$

and

$$U \equiv a_1 q + a_2 \phi(x)\, q + a_3 y q.$$

Therefore,

$$\begin{aligned} &Ux = 0; \\ &Uy = a_1 + a_2 \phi(x) + a_3 y, \\ &U^2 y = a_1 a_3 + a_2 a_3 \phi(x) + a_3^2 y, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &U^n y = a_1 a_3^{n-1} + a_2 a_3^{n-1} \phi(x) + a_3^n y. \end{aligned}$$

Hence the transformation T_a of this group is defined by

$$(1) \qquad \begin{aligned} x' &= x, \\ y' &= y\, e^{a_3} + \left(\frac{a_1 + a_2\,\phi(x)}{a_3}\right)(e^{a_3} - 1); \end{aligned}$$

and the transformation T_b by

$$(2) \qquad \begin{aligned} x'' &= x', \\ y'' &= y'\, e^{b_3} + \left(\frac{b_1 + b_2\,\phi(x)}{b_3}\right)(e^{b_3} - 1). \end{aligned}$$

Therefore, if $T_b\, T_a = T_c$,

$$c_1 = \frac{a_3 + b_3 + 2\,\kappa\,\pi\,i}{e^{a_3 + b_3} - 1} \left\{ e^{b_3}(e^{a_3} - 1)\frac{a_1}{a_3} + (e^{b_3} - 1) \right\},$$

$$c_2 = \frac{a_3 + b_3 + 2\,\kappa\,\pi\,i}{e^{a_3 + b_3} - 1} \left\{ e^{b_3}(e^{a_3} - 1)\frac{a_2}{a_3} + e^{b_3} - 1)\frac{b_2}{b_3} \right\},$$

$$c_3 = a_3 + b_3 + 2\,\kappa\,\pi\,i.$$

For finite values of the a's and b's, every branch of c_3 is finite, and at least one branch both of c_1 and of c_2 is finite. For c_1 and c_2 can only be infinite for $a_3 + b_3 = 2\,m\,\pi\,i$ (m an integer); but if $a_3 + b_3 = 2\,m\,\pi\,i$, the branches of c_1 and c_2 corresponding to $\kappa = -m$ are finite, being equal respectively to

$$\left[e^{b_3}(e^{a_3} - 1)\frac{a_1}{a_2} + (e^{b_3} - 1) \right]_{a_3 + b_3 = 2\,m\,\pi\,i} = e^{b_3}(e^{a_3} - 1)\left(\frac{a_1}{a_3} - 1\right),$$

and

$$\left[e^{b_3}(e^{a_3} - 1)\frac{a_2}{a_3} + (e^{b_3} - 1)\frac{b_2}{b_3} \right]_{a_3 + b_3 = 2\,m\,\pi\,i} = e^{b_3}(e^{a_3} - 1)\left(\frac{a_2}{a_3} - \frac{b_2}{b_3}\right) = e^{b_3}(e^{-b_3} - 1)\left(\frac{a_2}{a_3} - \frac{b_2}{b_3}\right).$$

The group

$$q, \quad \phi_2(x)\,q, \;\ldots\; \phi_{r-1}(x)\,q, \quad y\,q,$$

is likewise continuous for values of $r > 3$; i. e. for values of $r > 3$, it does not contain singular transformations. For, if $r = \rho > 3$ the transformations T_a are defined by

$$\begin{aligned} x' &= x, \\ y' &= y\, e^{a_\rho} + \left(\frac{a_1 + a_2\phi_2(x) + a_3\phi_3(x) + \ldots + a_{\rho-1}\phi_{\rho-1}(x)}{a_\rho}\right)(e^{a_\rho} - 1). \end{aligned}$$

Example IV.

$$e^{\alpha_\kappa x} q, \quad x e^{\alpha_\kappa x} q, \ldots, x^{\rho_\kappa} e^{\alpha_\kappa x} q, \quad y q, \quad p$$

$$\kappa = 1, 2, \ldots, m, \quad \alpha_\kappa = \text{const.}, \quad \Sigma \rho_\kappa + m = r - 2, \quad r > 3$$

Put $\rho_\kappa = 1$, $\kappa = m = 1$, and $r = 4$. We then have the group

$$e^{\alpha x} q, \quad x e^{\alpha x} q, \quad y q, \quad p;$$

and
$$U \equiv (a_1 e^{\alpha x} + a_2 x e^{\alpha x} + a_3 y) q + a_4 p.$$

Therefore,
$$U x = a_4, \qquad U^2 x = 0, \ldots U^n x = 0,$$

so that
$$x' = x + a_4;$$

and

$$U y = a_1 e^{\alpha x} + a_2 x e^{\alpha x} + a_3 y,$$

$$U^2 y = a_1 a_3 e^{\alpha x} + a_2 a_3 x e^{\alpha x} + a_3^2 y + e^{\alpha x}(a_1 a_4 \alpha + a_2 a_4) + x e^{\alpha x} a_2 a_4 \alpha,$$

$$U^3 y = a_1 a_3^2 e^{\alpha x} + a_2 a_3^2 x e^{\alpha x} + a_3^3 y + e^{\alpha x}(a_1 a_4 \alpha a_3 + a_2 a_4 a_3) + x e^{\alpha x} a_2 a_4 \alpha a_3 \\ + e^{\alpha x}(a_1 a_4^2 \alpha^2 + 2 a_2 a_4^2 \alpha) + x e^{\alpha x} a_2 a_4^2 \alpha^2,$$

$$U^4 y = a_1 a_3^3 e^{\alpha x} + a_2 a_3^3 x e^{\alpha x} + a_3^4 y + e^{\alpha x}(a_1 a_4 \alpha a_3^2 + a_2 a_4 a_3^2) + x e^{\alpha x} a_2 a_4 \alpha a_3^2 \\ + e^{\alpha x}(a_1 a_4^3 \alpha^3 + 3 a_2 a_4^3 \alpha^2) + x e^{\alpha x} a_2 a_4^3 \alpha^3 + e^{\alpha x}(a_1 a_4^2 \alpha^2 a_3 + 2 a_2 a_4^2 \alpha a_3) \\ + x e^{\alpha x} a_2 a_4^2 \alpha^2 a_3,$$

. . . .

Hence

$$y' = y e^{a_3} + x e^{\alpha x} \left\{ \frac{a_2}{\alpha a_4 - a_3} (e^{\alpha a_4} - e^{a_3}) \right\}$$

$$+ e^{\alpha x} \left[\frac{\{a_1(a_3 - \alpha a_4) + a_2 a_4\}(e^{a_3} - 1) - a_1(a_3 - \alpha a_4)(e^{\alpha a_4} - 1) - a_2 a_4 e^{\alpha a_4}(a_3 - \alpha a_4 + 1)}{(a_3 - \alpha a_4)^2} \right].$$

For the values of r, ρ_κ, and κ, chosen, this group is continuous.